Stability Analysis of Nonlinear Systems Using Lyapunov Theory

By: Nafees Ahmed
Outline

- Motivation
- Definitions
- Lyapunov Stability Theorems
- Analysis of LTI System Stability
- Instability Theorem
- Examples
References

- Dr. Radhakant Padhi, AE Dept., IISc-Bangalore (NPTEL)
- Control system, principles and design by M. Gopal, Mc Graw Hill
Techniques of Nonlinear Control Systems Analysis and Design

- **Phase plane analysis:** Up to 2\textsuperscript{nd} order or maxi 3\textsuperscript{rd} order system (graphical method)
- **Differential geometry** (Feedback linearization)
- **Lyapunov theory**
- **Intelligent techniques:** Neural networks, Fuzzy logic, Genetic algorithm etc.
- **Describing functions**
- **Optimization theory** (variational optimization, dynamic programming etc.)
Motivation

- Eigenvalue analysis concept does not hold good for nonlinear systems.
- Nonlinear systems can have multiple equilibrium points and limit cycles.
- Stability behaviour of nonlinear systems need not be always global (unlike linear systems). So we seek stability near the equilibrium point.
- Stability of non linear system depends on both initial value and its input (Unlike linear system). Stability of linear system is independent of initial conditions.
- Need of a systematic approach that can be exploited for control design as well.
Idea

- Lyapunov’s theory is based on the simple concept that the energy stored in a stable system can’t increase with time.
Definitions

System Dynamics

\[
\dot{X} = f(X) \quad f : D \rightarrow \mathbb{R}^n \text{ (a locally Lipschitz map)}
\]

\[
D : \text{an open and connected subset of } \mathbb{R}^n
\]

Equilibrium Point \((X_e)\)

\[
\dot{X}_e = f(X_e) = 0
\]

Note:

- Above system is an autonomous (i/p, u=0)
- Here Lyapunov stability is considered only for autonomous system (It can also extended to non autonomous system)
- We can have multiple equilibrium points
- We are interested in finding the stability at these equilibrium points
- \(\mathbb{R}^n \Rightarrow n\text{ dimensions (ie } x_1, x_2 => n=2 => \text{two dimensions })\)
Definitions

**Open Set:** Let set $A$ be a subset of $\mathbb{R}$ then the set $A$ is open if every point in $A$ has a neighborhood lying in the set. Or open set means boundary lines are not included. Mathematically

A set $A \subset \mathbb{R}^n$ is open if

for every $p \in A$, $\exists B_r(p) \subset A$

**Connected Set**

- A **connected set** is a set which cannot be represented as the union of two or more disjoint nonempty open subsets.
- Intuitively, a set with only one piece.
Definitions

- **Open set:**

  A set $A \subset \mathbb{R}^n$ is called as open, if for each $x \in A$ there exist an $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon)$ is contained in $A$. Such an interval is often called as $\varepsilon$-neighborhood of $x$ or simply neighborhood of $x$. 
Definitions

**Stable Equilibrium**

\( X_e \) is stable, provided for each \( \varepsilon > 0 \), \( \exists \delta(\varepsilon) > 0 \):

\[
\| X(0) - X_e \| < \delta(\varepsilon) \quad \Rightarrow \quad \| X(t) - X_e \| < \varepsilon \quad \forall t \geq t_0
\]

**Unstable Equilibrium**

If the above condition is not satisfied, then the equilibrium point is said to be unstable
1. Starting with a small ball of radius $\delta(\varepsilon)$ from initial condition $X_0$, a system will move anywhere around the ball but will not leave the ball of radius $\varepsilon$.
2. Ball $\delta(\varepsilon)$ is a function of $\varepsilon$.
3. Size of $\delta(\varepsilon)$ may be larger than the ball of radius $\varepsilon$. 
Definitions

**Convergent Equilibrium**

If \( \exists \delta : \|X(0) - X_e\| < \delta \implies \lim_{t \to \infty} X(t) = X_e \)
**Convergent system**: Starting from any initial condition $X_0$, system may go anywhere but finally converges to equilibrium point $X_e$.
Definitions

**Asymptotically Stable**
If an equilibrium point is both stable and convergent, then it is said to be asymptotically stable.

**Note:** System will never leave the $\epsilon$ bound and finally will converge to equilibrium point $X_e$. 
Definitions

\[ Z = X - X_e \Rightarrow \dot{Z} = \dot{X} - \dot{X}_e \Rightarrow \dot{Z} = \dot{X} (X_e = 0) = f(Z) \Rightarrow \]

**Exponentially Stable**

\[ \exists \alpha, \lambda > 0 : \|X(t) - X_e\| \leq \alpha \|X(0) - X_e\| e^{-\lambda t} \quad \forall t > 0 \]

whenever \[ \|X(0) - X_e\| < \delta \]

**Convention**

The equilibrium point \( X_e = 0 \) (without loss of generality)
Definitions

A scalar function $V : D \rightarrow \mathbb{R}$ is said to be

- **Positive definite function**: if following conditions are satisfied
  
  \begin{align*}
  (i) & \quad 0 \in D \quad \text{and} \quad V(0) = 0 \\
  (ii) & \quad V(X) > 0 \quad \text{in} \quad D - \{0\}
  \end{align*}
  
  (domain $D$ excluding 0)

- **Positive semi definite function**: 
  
  \begin{align*}
  (i) & \quad 0 \in D \quad \text{and} \quad V(0) = 0 \\
  (ii) & \quad V(X) \geq 0, \quad \forall X \in D
  \end{align*}

- **Negative define function**: (i) condition same, (ii) $\leq$

- **Negative semi define function**: (i) condition same, (ii) $\leq$

**Note:**

1. Output of function $V(x)$ is a scalar value, hence $V(x)$ is scalar function.
2. Negative define (semi definite) if $-V(x)$ is + definite (semi definite)
Lyapunov Stability Theorems

Theorem – 1 (Stability)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \rightarrow \mathbb{R}^n$.
Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$
(ii) $V(X) > 0$, in $D - \{0\}$
(iii) $\dot{V}(X) \leq 0$, in $D - \{0\}$

Then $X = 0$ is "stable".

Note:
Condition (i) & (ii) $\Rightarrow V(X)$ positive definite
Condition (iii) $\Rightarrow \dot{V}(X)$ Negative semi definite
What about $V(X)$

- There is no general method for selection of $V(X)$.
- Some time select $V(X)$ such that its properties are similar to energy i.e.
  \[ V(X) = \frac{1}{2} X^T X \]

- Or $V(X) = \text{Kinteic Energy} + \text{Potential Engery}$

- Or $V(X) = x_1^2 + x_2^2 \text{ etc}$

- How to calculate $\dot{V}(X)$

  \[
  \dot{V}(X) = \left( \frac{\partial V}{\partial x} \right)^T \dot{x} = \left( \frac{\partial V}{\partial x} \right)^T f(X)
  \]
Lyapunov Stability Theorems

Theorem – 2 (Asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X), \ f : D \rightarrow \mathbb{R}^n$.
Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$
(ii) $V(X) > 0, \text{ in } D - \{0\}$
(iii) $\dot{V}(X) < 0, \text{ in } D - \{0\}$

Then $X = 0$ is "asymptotically stable".

Note:
Condition (i) & (ii) $\Rightarrow V(X) \text{ positive definite}$
Condition (iii) $\Rightarrow \dot{V}(X) \text{ Negative definite}$
Lyapunov Stability Theorems

Theorem – 3 (Globally asymptotically stable)
Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \to \mathbb{R}^n$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that:

(i) $V(0) = 0$
(ii) $V(X) > 0$, in $D - \{0\}$
(iii) $\dot{V}(X) < 0$, in $D - \{0\}$
(iv) $V(X)$ is "radially unbounded"

Then $X = 0$ is "globally asymptotically stable".
Radially Unbounded?

The more and more you go away from the equilibrium point, $V(X)$ will increase more and more.
Lyapunov Stability Theorems

Theorem – 3 (Exponentially stable)

Suppose all conditions for asymptotic stability are satisfied. In addition to it, suppose \( \exists \) constants \( k_1, k_2, k_3, p \):

(i) \( k_1 \|X\|^p \leq V(X) \leq k_2 \|X\|^p \)

(ii) \( \dot{V}(X) \leq -k_3 \|X\|^p \)

Then the origin \( X = 0 \) is "exponentially stable". Moreover, if these conditions hold globally, then the origin \( X = 0 \) is "globally exponentially stable".

Note: Global \( \Rightarrow \) Subset D = R
Example:

Pendulum Without Friction

- System dynamics
  \[
  \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
  \end{bmatrix} = \begin{bmatrix}
  x_2 \\
  -(g/l) \sin x_1
  \end{bmatrix}
  \]

- Lyapunov function
  \[
  V = KE + PE = \frac{1}{2} m (\omega l)^2 + mgh = \frac{1}{2} ml^2 x_2^2 + mg(1 - \cos x_1)
  \]
Pendulum Without Friction

\[ \dot{V}(X) = (\nabla V)^T f(X) \]
\[ = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) \\ f_2(X) \end{bmatrix}^T \]
\[ = \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{l} \sin x_1 \end{bmatrix}^T \]
\[ = mglx_2 \sin x_1 - mglx_2 \sin x_1 = 0 \]
\[ \dot{V}(X) \leq 0 \quad \text{(nsdf)} \]

Hence, it is a “stable” system.
Pendulum With Friction

Modify the previous example by adding the friction force $kl \dot{\theta}$

$$ma = -mg \sin \theta - kl \dot{\theta}$$

Defining the same state variables as above

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2$$
Pendulum With Friction

\[ \dot{V}(X) = (\nabla V)^T f(X) \]

\[ = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T \]

\[ = \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}^T \]

\[ = -kl^2 x_2^2 \]

\[ \dot{V}(X) \leq 0 \quad \text{(nsdf)} \]

Hence, it is also just a “stable” system.
(A frustrating result...)
NOTE

- Here, pendulum with friction should be asymptotically stable as it comes to an equilibrium point finally due to friction ($\Rightarrow \dot{V}(X)$ should be negative definite not negative semi definite nsdf)
- But we are not able to prove this.
- Because
  - when $x_2 \neq 0$, $\dot{V}(X)$ will always be $-Ve$
  - But when $x_2 = 0$ There are multiple equilibrium points on $x_1$ line.
  - Negative definite means the movement I go away from the zero I should get $-ve$ value
Example:

Consider the system described by the equations

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_1 - x_2^3 \]

Solution:

Choose \( V(x) = x_1^2 + x_2^2 \)

Which satisfies following two conditions that is it is positive definite

\[ V(0) = 0 \& V(x) > 0 \]
\[ \dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 = 2x_1 \dot{x}_1 + 2x_2 (-x_1 - x_2^3) = -2x_2^4 \]
\[ \dot{V}(x) \leq 0 \Rightarrow \text{nsdf (similar to pendulum with friction)} \]

So system is stable, we can’t say asymptotically stable
Analysis of LTI system using Lyapunov stability

System dynamics: \[ \dot{X} = AX, \quad A \in \mathbb{R}^{n \times n} \]

Lyapunov function: \[ V(X) = X^T PX, \quad P > 0 \quad (pdf) \]

Derivative analysis:
\begin{align*}
\dot{V} &= \dot{X}^T PX + X^T P\dot{X} \\
&= X^T A^T PX + X^T PAX \\
&= X^T (A^T P + PA) X
\end{align*}

Note: \[ \dot{X} = AX \Rightarrow \dot{X}^T = (AX)^T = X^T A^T \]
Analysis of LTI system using Lyapunov stability...

For stability, we aim for
\[ \dot{V} = -X^T Q X \quad (Q > 0) \]

By comparing
\[ X^T \left( A^T P + PA \right) X = -X^T Q X \]

For a non-trivial solution
\[ PA + A^T P + Q = 0 \]

(Lyapunov Equation)
Analysis of LTI system using Lyapunov stability….

**Theorem:** The eigenvalues $\lambda_i$ of a matrix $A \in \mathbb{R}^{n \times n}$ satisfy $\text{Re}(\lambda_i) < 0$ if and only if for any given symmetric pdf matrix $Q$, $\exists$ a unique pdf matrix $P$ satisfying the Lyapunov equation.
Step to solve
Analysis of LTI system using Lyapunov stability….

- Choose an arbitrary symmetric positive definite matrix $Q$ ($Q = I$)
- Solve for the matrix $P$ form the Lyapunov equation and verify whether it is positive definite
- Result: If $P$ is positive definite, then $\dot{V}(X) < 0$ and hence the origin is “asymptotically stable”.
Example: Analysis of LTI system using Lyapunov stability

- Determine the stability of the system described by the following equation
- \( \dot{x} = Ax \) with \( A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \)
- Solution:
  - \( A^T P + PA = -Q = -I \)
  - \( \begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \)
- Note here we took \( p_{12} = p_{21} \) because Matrix P will be positive real symmetric matrix
- \( -2p_{11} + 2p_{12} = -1 \)
- \( -2p_{11} - 5p_{12} + p_{22} = 0 \)
- \( -4p_{12} - 8p_{22} = -1 \)

Solving above three equations \( P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \frac{23}{60} & -\frac{7}{60} \\ -\frac{7}{60} & \frac{11}{60} \end{bmatrix} \)

which is seen to be positive definite. Hence this system is asymptotically stable.
Till now?

- All were Lyapunov Direct methods
- There are some indirect methods also
Lyapunov’s Indirect Theorem

Let the linearized system about $X = 0$ be $\Delta \dot{X} = A(\Delta X)$. The theorem says that if all the eigenvalues $\lambda_i$ ($i = 1, \ldots, n$) of the matrix $A$ satisfy $\text{Re}(\lambda_i) < 0$ (i.e. the linearized system is exponentially stable), then for the nonlinear system the origin is locally exponentially stable.
**Instability theorem**

Consider the autonomous dynamical system and assume $X=0$ is an equilibrium point. Let $V : D \to \mathbb{R}$ have the following properties:

(i) $V(0) = 0$

(ii) $\exists X_0 \in \mathbb{R}^n$, arbitrarily close to $X = 0$, such that $V(X_0) > 0$

(iii) $\dot{V} > 0$ $\forall X \in U$, where the set $U$ is defined as follows

$U = \{X \in D : \|X\| \leq \varepsilon$ and $V(X) > 0\}$

Under these conditions, $X=0$ is unstable
In rough way

- In rough way instability theorem state that
  - if $V(X)$ positive definite
  - then $\dot{V}(X)$ should also be positive definite
Positive Definite Matrices

Definition: Symmetric matrix \( M = M^T \) is

- **positive definite** \((M > 0)\) if \( x^T M x > 0, \forall x \neq 0 \)
- **positive semidefinite** \((M \geq 0)\) if \( x^T M x \geq 0, \forall x \)
Thanks?