Stability Analysis of Nonlinear System by Popov's Stability Criterion

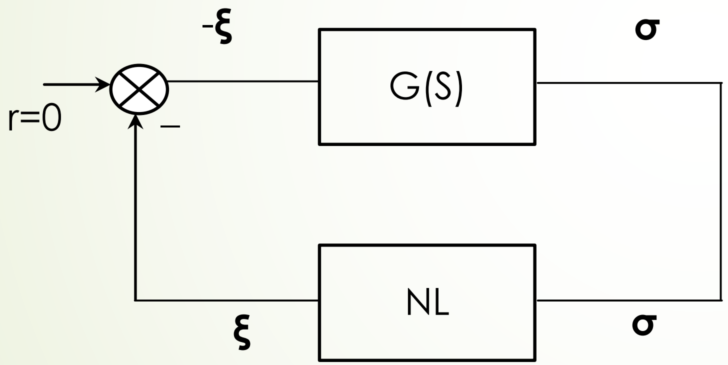
By: Nafees Ahmed

Introduction

- It is the first technique given for nonlinear systems analysis. (1962)
- It is in frequency domain technique.
- It can be seen in several view points.
- We will see it with loop transformation

Consider the following closed loop system

- Many nonlinear physical systems can be represented as a feedback connection of a linear dynamical system and a nonlinear element.



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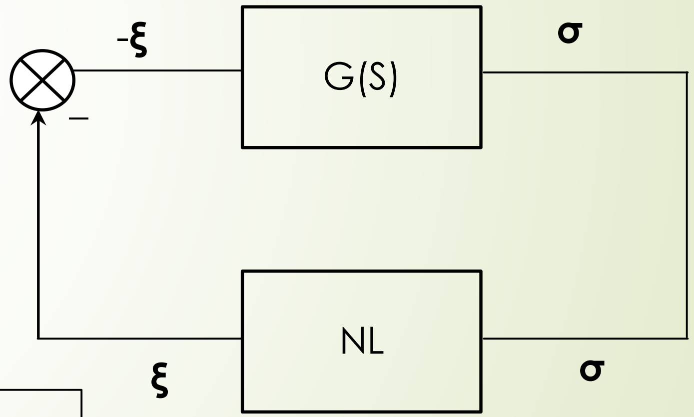
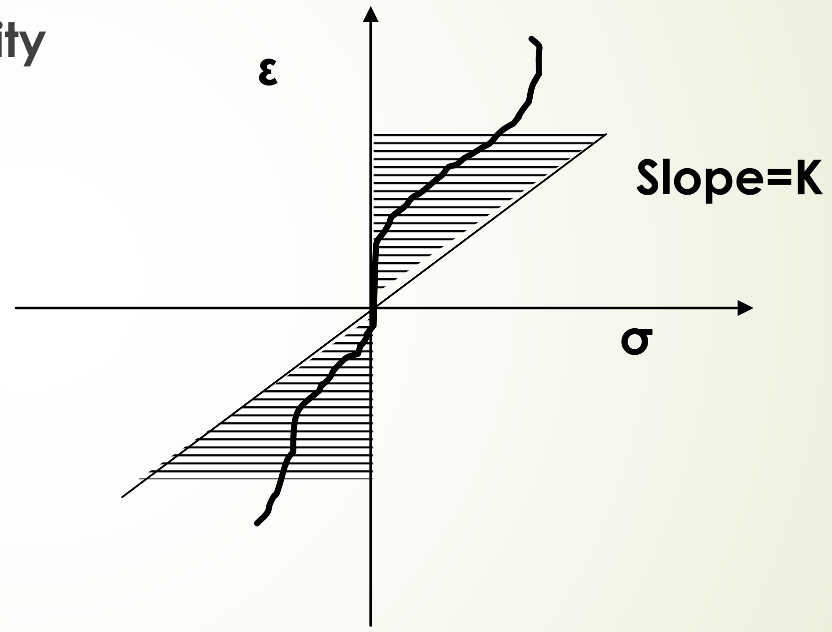


T.F. of linear system
G(s)=σ/-ε

NL=> Non linear System

Consider following nonlinearity

- Let us define the nonlinearity
- NL ϵ m[K, ∞) sector

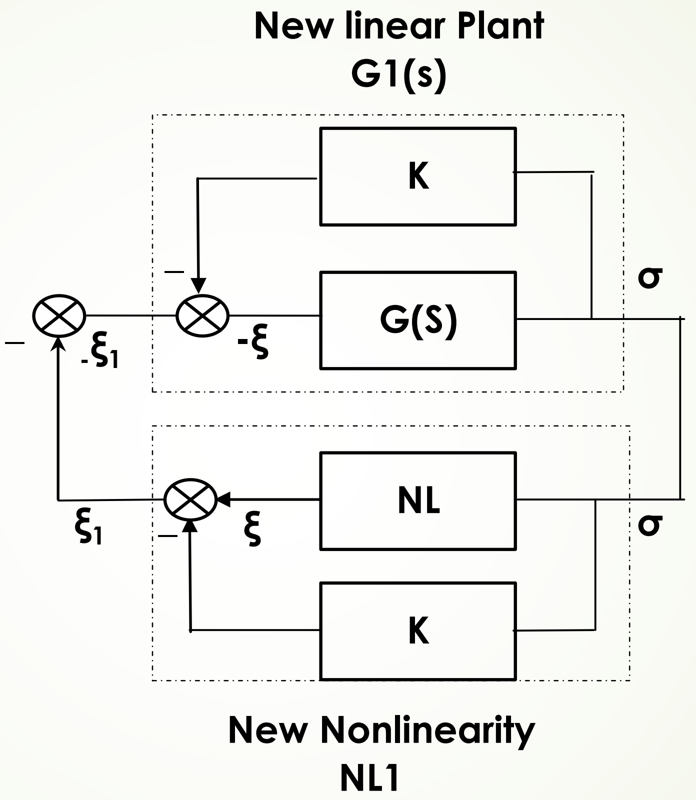


- Above nonlinearity can be transformed to new nonlinearity NL1 ϵ m[0, ∞] sector

Loop transformation

- Above sys can be transformed to a new system as shown on next slide keeping same i/p but different o/p for non linear part.
- T.F of new linear plat $G1(s) = \frac{\sigma}{-\xi 1} = \frac{\sigma}{-\xi + K\sigma} = \frac{G(s)}{1+KG(s)}$
- In this way nonlinearity NL is transformed to NL1 i.e. From m[K, ∞) sector to m[0, ∞) sector
- Note: here i/p & o/p relationship of G(s) and NL are same as the original one.

Loop transformation …



Where
$\xi_1 = \xi - \sigma K$

Nonlinearity of NL1 ϵ m[0, ∞) sector

Loop transformation …

- If a transfer function is Passive => it is positive real

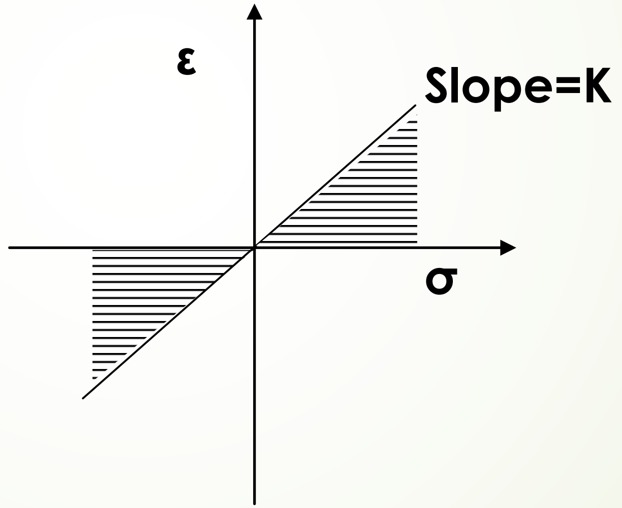
  => it is asymptotic stable
- Interconnection of two passive systems is also passive system.

Loop transformation …

- Now we can say that if original system is asymptotic stabile then its transformed sys shown on previous slide will also be asymptotic stable.
- New real plant G1(s) must be positive real i.e $G1(s) = \frac{G(s)}{1+KG(s)}$ must be a positive real.
- New nonlinearity defined by NL1 ϵ m[0, ∞) sector

Loop transformation…

- Same way we can do other loop transformation
- Consider following nonlinearity
- NL ϵ m[0, K] sector

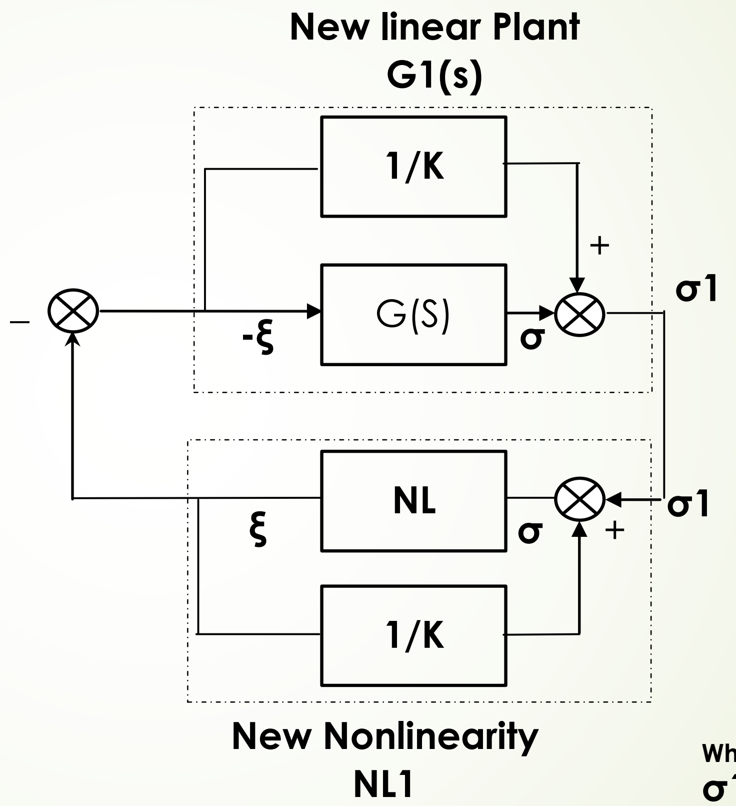


- Above nonlinearity can be transformed to new nonlinearity NL1 ϵ m[0, ∞) sector

Loop transformation…

- Above sys can be transformed to a new system as shown on next slide keeping same o/p but different i/p for non linear part.
- T.F of new linear plat $G1(s)$ =G(s)+1/K
- In this way nonlinearity NL is transformed to NL1 i.e. From m[0,K] sector to m[0, ∞) sector
- Note: here i/p & o/p relationship of G(s) and NL are same as the original one.

Loop transformation…



Where
$\sigma 1 = \sigma - (1/K)\xi$

Nonlinearity of NL1 ϵ m[0, ∞) sector

Loop transformation …

- Similar to previous
- Now we can say that if original system is asymptotic stabile then its transformed sys shown on previous slide will also be asymptotic stable.
- New real plant G1(s) must be positive real i.e $G1(s) = G(s) + 1/K$ must be a positive real.
- New nonlinearity defined by NL1 ϵ m[0, ∞) sector

Loop transformation …

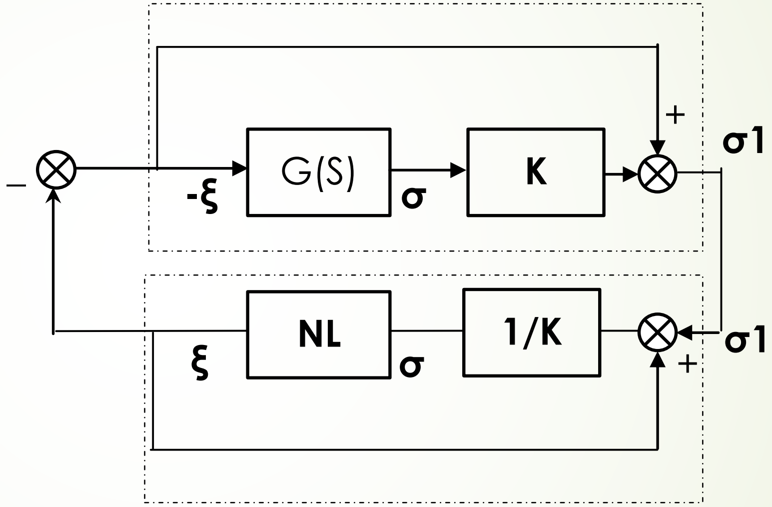
- Note: In above two loop transformation one portion is feed forward and other portion is feed backward with same sign.
- In above loop transformation we introduced a constant K (=slop of nonlinearity), to change the nonlinearity.

Loop transformation …

- With same nonlinearity NL ϵ m[0, K] sector
- The above loop transformation was done in feedback and feed forward loop, it can also be done in forward path only as shown on next slide

Loop transformation …

New linear Plant
G1(s)



New Nonlinearity
NL1

Where
$\sigma 1 = K\sigma - \xi$

Nonlinearity of NL1 ϵ m[0, ∞) sector

Loop transformation …

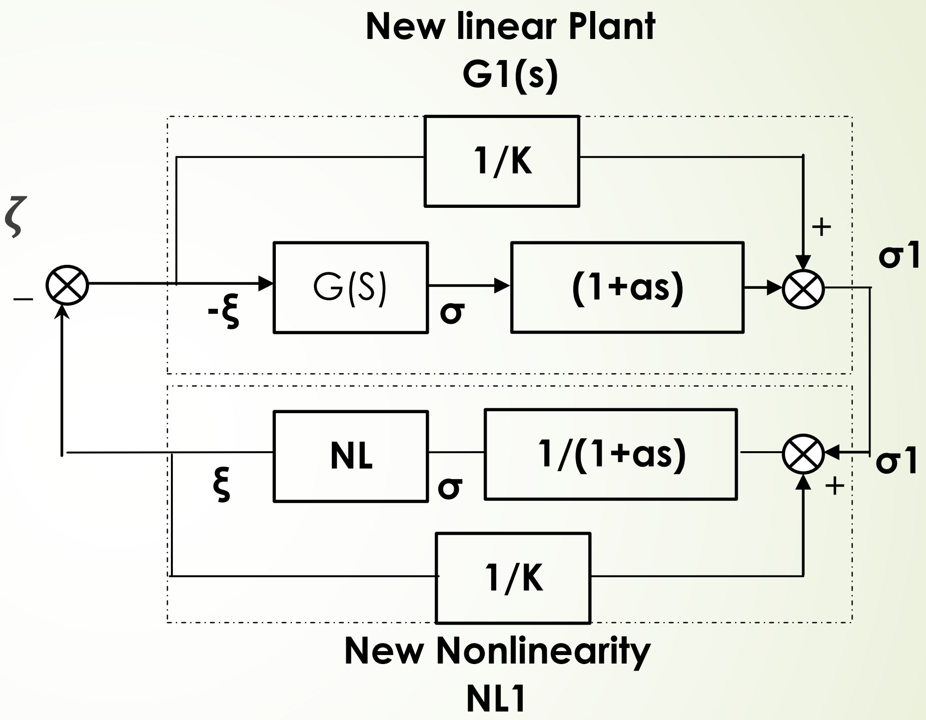
- Similar to previous
- Now we can say that if original system is asymptotic stabile then its transformed sys shown on previous slide will also be asymptotic stable.
- New real plant G1(s) must be positive real i.e$G1(s) = 1 + \mathrm{KG}(\mathrm{s})$ must be a positive real.
- New nonlinearity defined by NL1 ϵ m[0, ∞) sector

Loop transformation …

- Now use mix of above two methods
- And instead of K we can also use some transfer function
- With same nonlinearity NL ϵ m[0, K] sector

Loop transformation …

- $Where\ \sigma 1 = a\dot{\sigma} + \sigma - \frac{1}{K}\zeta$



Nonlinearity of NL1 ϵ m[0, ∞) sector

Loop transformation …

- Similar to previous
- Now we can say that if original system is asymptotic stabile then its transformed sys shown on previous slide will also be asymptotic stable.
- New real plant G1(s) must be positive real i.e $G1(s) = (1 + \mathrm{as})\mathrm{G}(\mathrm{s}) + 1/\mathrm{K}$ must be a positive real.

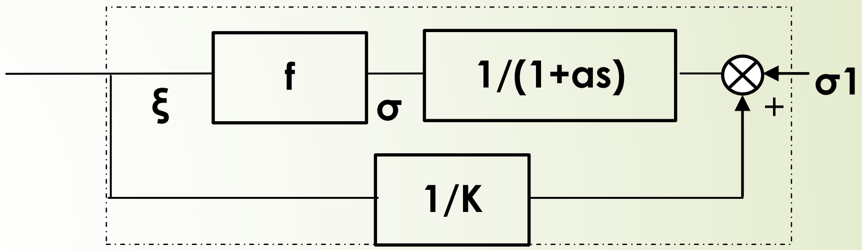
Interconnection of 2 passive systemes

- If G1(s) is passive and NL1 is passive then there interconnection will also be passive. (Passive=>positive real)
- So passivity for new linear system G1(s) $\dot{V}1(X) \leq -\sigma 1 \xi$
- We have to prove that new nonlinear system NL1 will also be passive. So that overall interconnection will also be passive.
- For this make the differential equation of NL1 assuming NL=f as shown on next slide

Note: We know Lyapunov function of a passive system will have $\dot{V}(X) \leq uy$ where u = i/p & y=o/p (passivity in term of Lyapunov function)

Interconnection of 2 passive systemes…

- Let $\xi = f(\sigma)$ & f ϵ m[0, K] sector
- $\sigma 1 = a\dot{\sigma} + \sigma - \frac{1}{K}\xi$



New Nonlinearity
NL1

- $\Rightarrow \sigma 1 = a\dot{\sigma} + \sigma - \frac{1}{K}f(\sigma)$
- Now take σ as state and $\sigma 1$ as input
- So we can write the state equ

$a\dot{\sigma} = -\sigma + \frac{1}{K}f(\sigma) + \sigma 1$ state equ

$\xi = f(\sigma)$ output equ

Interconnection of 2 passive systemes…

- **Take σ=x and ξ=y & σ1=u sate equ**

  $\boldsymbol{a\dot{x} = -x + \frac{1}{K} f(x) + u}$ **state equ**

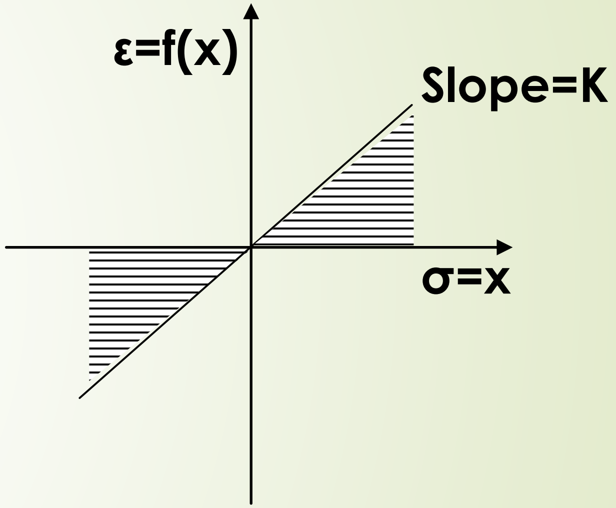
  y= $\boldsymbol{f(x)}$ **output equ**

Let us define a Laypunov function for NL1 as

$\boldsymbol{V_2(x) = a\int_0^x f(x)dx}$ **which is always >0** $\forall \boldsymbol{x}$

$$\Rightarrow \boldsymbol{\dot{V}_2(x) = af(x)\dot{x}}$$

- Put the value of $\boldsymbol{a\dot{x}}$ from above equation
- $\boldsymbol{\dot{V}_2(x) = -f(x)x + \frac{1}{K} f(x)^2 + uf(x)}$
- So $\boldsymbol{\dot{V}_2(x) \le uf(x) \le uy}$ $\Rightarrow$ NL1 is also passive



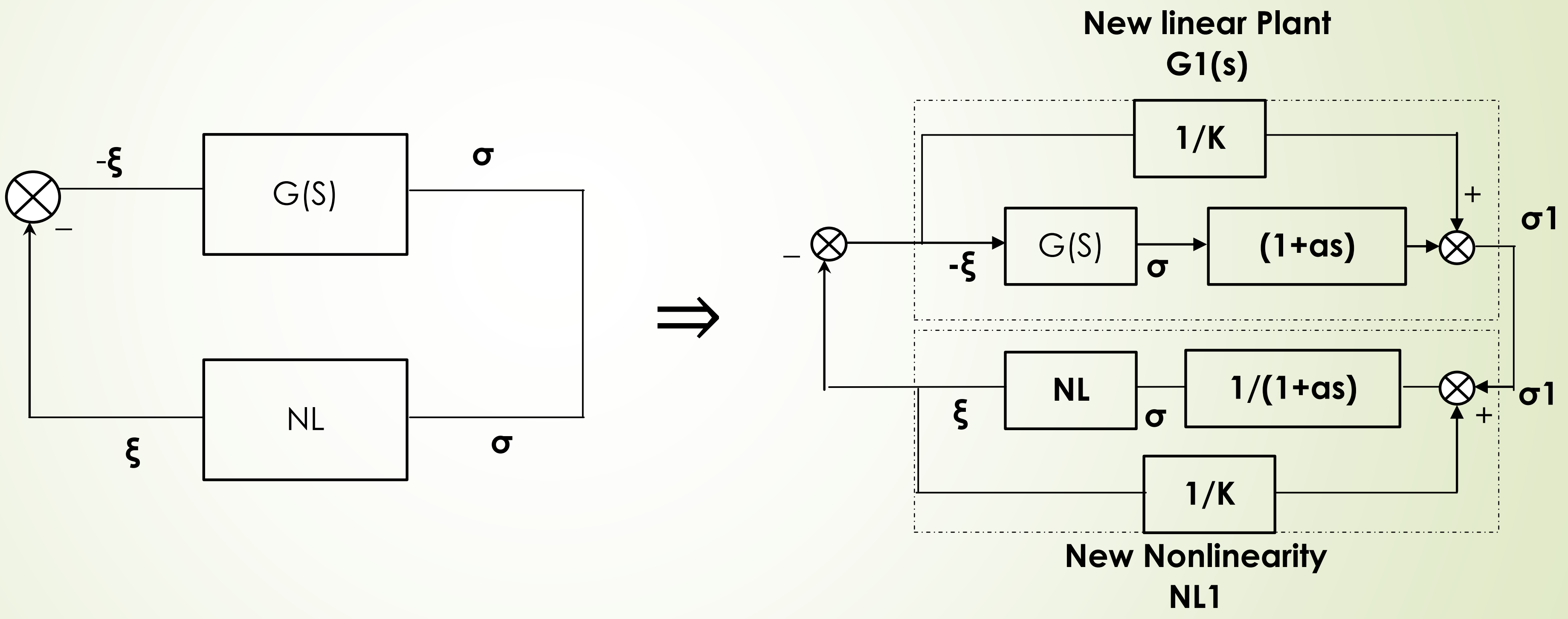
Note: $\boldsymbol{\int_0^x f(x)dx}$ is always +ve for above non linearity

Interconnection of 2 passive systemes…

- **We know Lyapunov's function of new linear system (Passive) G1(s)**
  - **$V1(x)=x^TPx$ with $\dot{V}1(X) \leq -\sigma 1\xi$ P is positive definite matrix**
- **And Lyapunov's function of new nonlinear system (Passive) NL1**
  - $V_2(x) = a\int_0^x f(x)dx \quad with \quad \dot{V}2(X) \leq \sigma 1\xi$
- **So Lyapunov's function of overall system**
  - **$V=V1+V2= x^TPx+a\int_0^x f(x)dx$ with $\dot{V} = \dot{V}1+\dot{V}2 \leq -\sigma 1\xi + \sigma 1\xi \leq 0$**
- **Here V>0 and $\dot{V} \leq 0$ and by Lyapunov this is the condition of Asymptotic stability**

Popov’s criterion

- Consider a system shown in following figure



Nonlinearity of NL1 ϵ m[0, K] sector

Nonlinearity of NL1 ϵ m[0, ∞) sector

Popov's criterion …

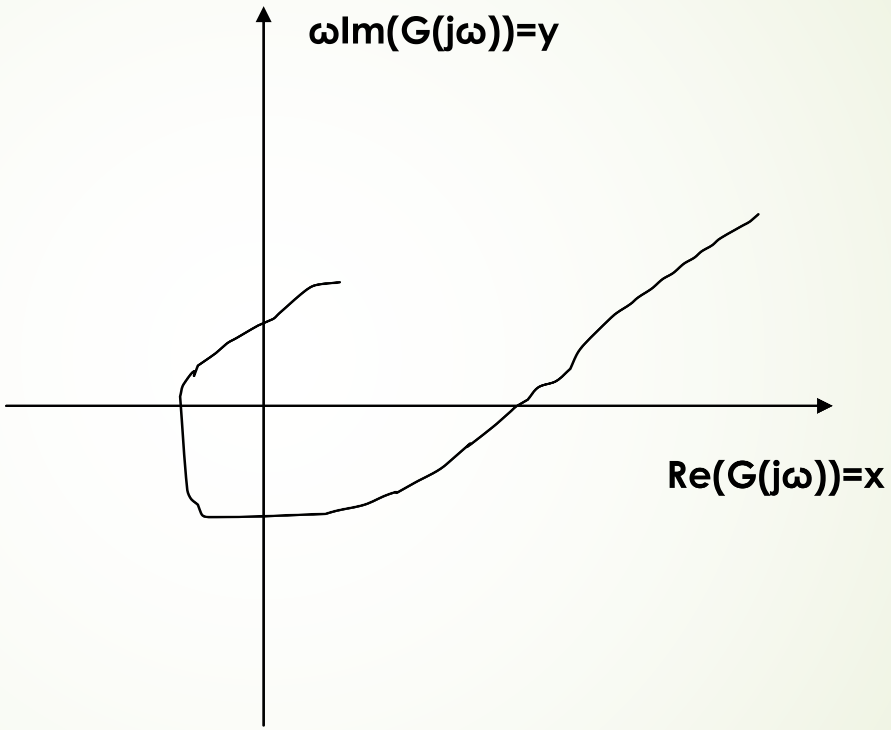
- This system will be asymptotically stable
- If $G1(s) = (1 + as)G(s) + 1/K$ is positive real i.e

  Re(G1(s)≥0 $\Rightarrow \mathrm{Re}[(1 + as)G(s) + 1/K$] ≥0
- For this G1(s) must be strictly proper [i.e. |deg(num)-deg(den)|≤1], necessary condition

Popov's criterion …

- How to check condition $\mathrm{Re}[(1 + as)G(s) + 1/K\,] \geq 0$
- => $\mathrm{Re}[(1 + ja\omega)(ReG(j\omega) + jI\mathrm{mG}(\mathrm{j}\omega))] + 1/K \geq 0$
- => $ReG(j\omega) - a\omega I\mathrm{mG}(\mathrm{j}\omega) + 1/K \geq 0$
- **For above draw the following plot called Popov's plot**

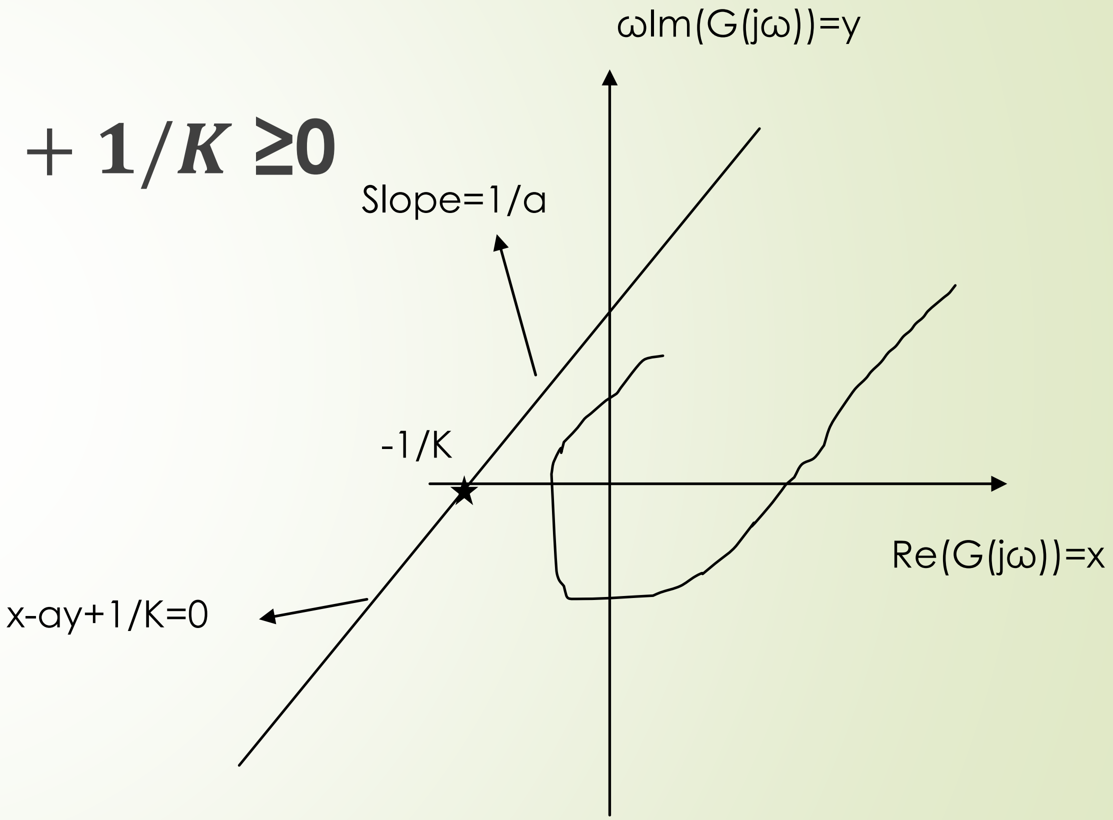
Popov's criterion …



Popov's Plot

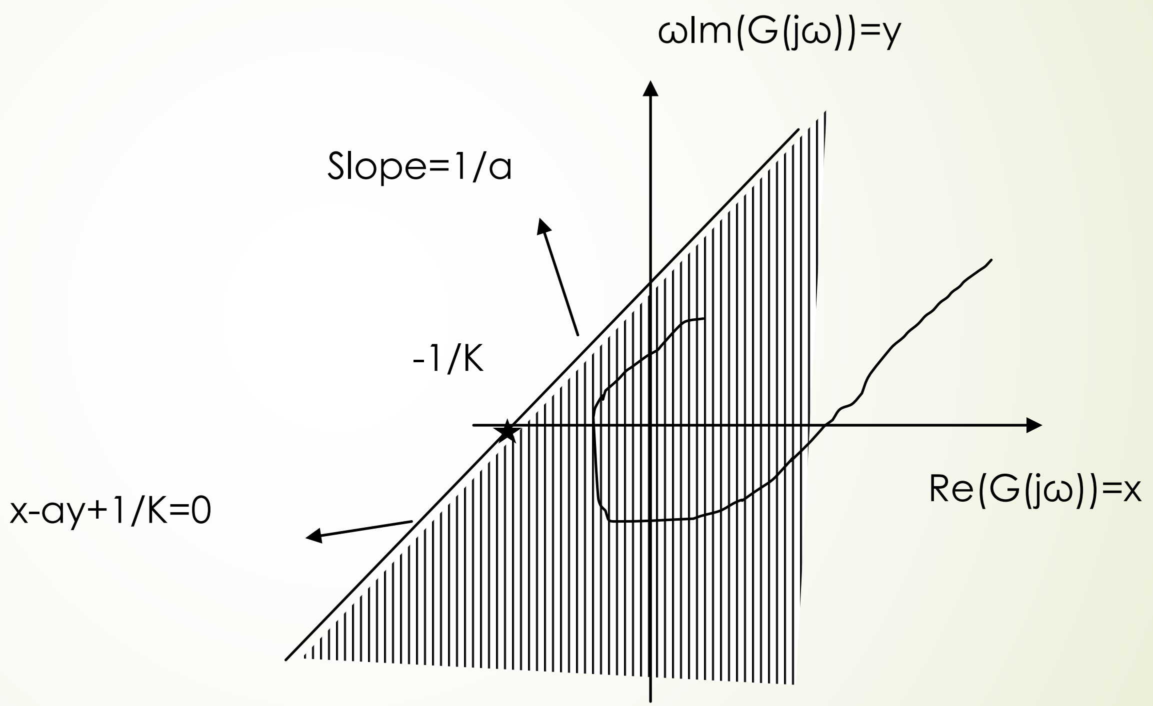
Popov's criterion …

- $ReG(j\omega) - a\omega ImG(j\omega) + 1/K \geq 0$
- $\Rightarrow x - ay + 1/K \geq 0$



Popov's Plot

Popov's criterion …



Popov's Plot

Popov's criterion …

- Hence to check the condition $\mathrm{Re}[(1 + as)G(s) + 1/K]$ ≥0 just draw Popov's plot and draw the line as shown above, if Popov's plot is to the right (below) of line that means above condition is satisfied. So the overall system will be asymptotically stable.
- **Note:** Difference b/w Popov's plot and Nyquist plot

Popov's is Re(G(jω) Vs ωIm(G(jω)

Nyquist plot is Re(G(jω) Vs Im(G(jω)

Reference

- **Prof. Harish K Pillai, IIT Bombay, through NPTEL**

Thanks

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