Stability Analysis of Nonlinear Systems Using Lyapunov Theory

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Outline

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- Lyapunov Stability Theorems
- Analysis of LTI System Stability
- Instability Theorem
- Examples

References

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- H. K. Khalil: *Nonlinear Systems*, Prentice Hall, 1996.
- H. J. Marquez: *Nonlinear Control Systems analysis and Design*, Wiley, 2003.
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- Control system, principles and design by M. Gopal, Mc Graw Hill

Techniques of Nonlinear Control Systems Analysis and Design

- **Phase plane analysis:** Up to 2nd order or maxi 3rd order system (graphical method)
- **Differential geometry** (Feedback linearization)
- **Lyapunov theory**
- **Intelligent techniques:** Neural networks, Fuzzy logic, Genetic algorithm etc.
- **Describing functions**
- **Optimization theory** (variational optimization, dynamic programming etc.)

Motivation

- Eigenvalue analysis concept does not hold good for nonlinear systems.
- Nonlinear systems can have multiple equilibrium points and limit cycles.
- Stability behaviour of nonlinear systems need not be always global (unlike linear systems). So we seek stability near the equilibrium point.
- Stability of non linear system depends on both initial value and its input (Unlike liner system). Stability of linear system is independent of initial conditions.
- Need of a systematic approach that can be exploited for control design as well.

Idea

- Lyapunov’s theory is based on the simple concept that the energy stored in a stable system can’t increase with time.

Definitions

System Dynamics

$$\dot{X} = f(X) \qquad f : D \rightarrow \mathbb{R}^n \ \text{(a locally Lipschitz map)}$$

$$D : \text{an open and connected subset of } \mathbb{R}^n$$

Equilibrium Point (X_e)

$$\dot{X}_e = f(X_e) = 0$$

Note:

- Above system is an autonomous (i/p, u=0)
- Here Lyapunov stability is considered only for autonomous system (It can also extended to non autonomous system)
- We can have multiple equilibrium points
- We are interested in finding the stability at these equilibrium points
- R^n => n dimensions (ie x_1,x_2 =>n=2 =>two dimensions)

Definitions

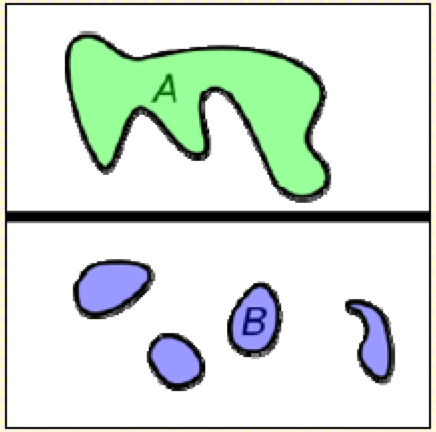
Open Set: Let set A be a subset of R then the set A is open if every point in A has a neighborhood lying in the set. Or open set means boundary lines are not included. Mathematically

A set $A \subset \mathbb{R}^n$ is open if

for every $p \in A,\ \exists\, B_r(p) \subset A$

Connected Set

- A **connected set** is a set which cannot be represented as the union of two or more disjoint nonempty open subsets.
- Intuitively, a set with only one piece.



Space *A* is connected, *B* is not.

Definitions

- **Open set:**
- A set $A \subset \mathbb{R}^n$ is called as open, if for each $x \in A$ there exist an $\varepsilon > 0$ such that the interval $(x - \varepsilon, x + \varepsilon)$ is contained in A. Such an interval is often called as ε -neighborhood of x or simply neighborhood of x.

Definitions

Stable Equilibrium

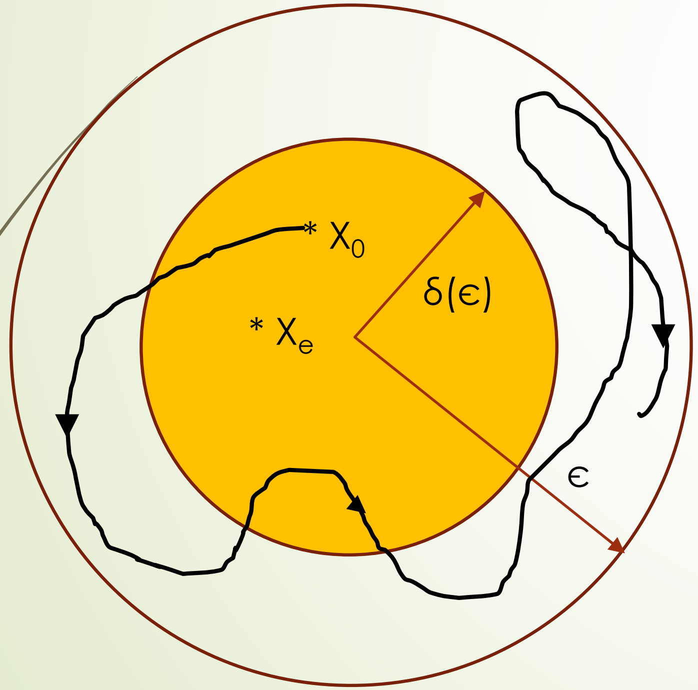
X_e is stable, provided for each $\varepsilon > 0,\ \exists \delta(\varepsilon) > 0:$

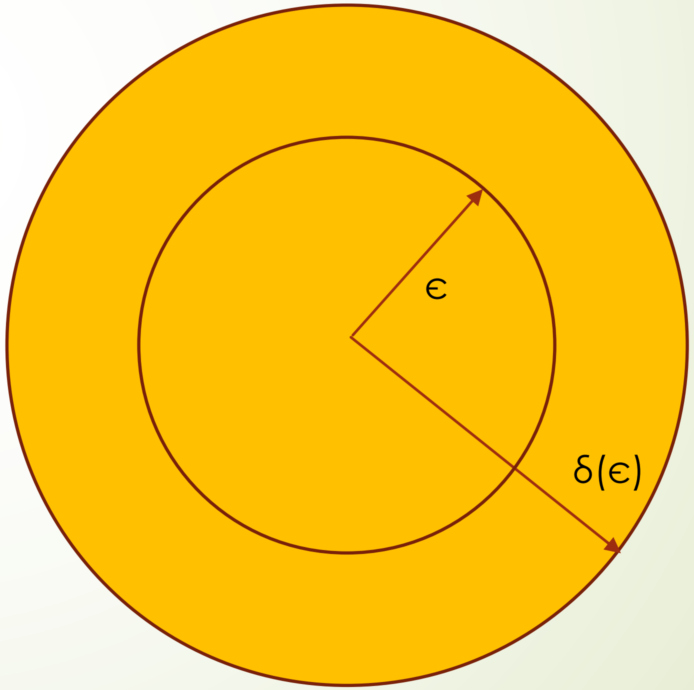
$$\|X(0) - X_e\| < \delta(\varepsilon) \quad \Rightarrow \quad \|X(t) - X_e\| < \varepsilon \quad \forall t \geq t_0$$

Unstable Equilibrium

If the above condition is not satisfied, then the equilibrium point is said to be unstable

1. Starting with a small ball of radius δ(ε) from initial condition X_o a system will move anywhere around the ball but will not leave the ball of radius ε
2. Ball δ(ε) is a function of ε.
3. Size of δ(ε) may be larger then ball of radius ε



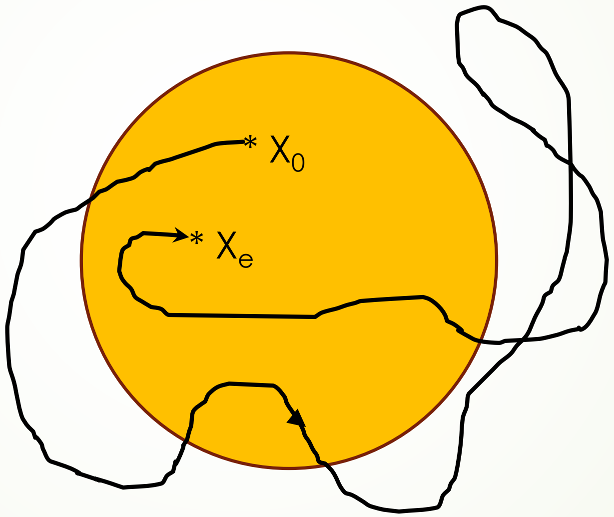


Definitions

Convergent Equilibrium

$$\text{If } \exists\delta: \quad \left\| X(0) - X_e \right\| < \delta \quad \Rightarrow \quad \lim_{t\to\infty} X(t) = X_e$$

Convergent system: Starting from any initial condition X_o, system may go anywhere but finally converges to equilibrium point X_e



Definitions

Asymptotically Stable

If an equilibrium point is both stable and convergent, then it is said to be asymptotically stable.

Note: System will never leave the ε bound and finally will converge to equilibrium point X_e.

Definitions

Exponentially Stable

$$\exists\, \alpha, \lambda > 0 : \quad \|X(t) - X_e\| \le \alpha \|X(0) - X_e\| e^{-\lambda t} \quad \forall t > 0$$

whenever $\|X(0) - X_e\| < \delta$

Convention

$$\boxed{\text{The equilibrium point } X_e = 0}$$

(without loss of generality)

Conversion : $Z = X - X_e \Rightarrow \dot{Z} = \dot{X} - \dot{X}_e \Rightarrow \dot{Z} = \dot{X}\ (X_e = 0) = f(Z) \Rightarrow$
$\dot{Z} = f(Z + X_e)$

Definitions

A scalar function V : D→R is said to be

- **Positive definite function:** if following condition are satisfied
  $$(i)\ 0 \in D \ \ and \ \ V(0) = 0$$
  $$(ii)\ V(X) > 0 \ \ in \ \ D - \{0\}$$
  (domain D excluding 0)
- **Positive semi definite function:**
  $$(i)\ 0 \in D \ \ and \ \ V(0) = 0$$
  $$(ii)\ V(X) \geq 0, \ \ \forall X \in D$$
- **Negative define function:** (i) condition same, (ii) <
- **Negative semi define function:** (i) condition same, (ii) ≤

Note:

1. Output of function V(x) is a scalar value, hence V(x) is scalar function .
2. Negative define (semi definite) if –V(x) is + definite (semi definite)

Lyapunov Stability Theorems

Theorem – 1 (Stability)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i)\quad V(0) = 0$

$(ii)\quad V(X) > 0,\ in\ D - \{0\}$

$(iii)\quad \dot{V}(X) \le 0,\ in\ D - \{0\}$

Then $X = 0$ is "stable".

Note:

Condition (i) & (ii) $\Rightarrow$ V(X) positive definite

Condition (iii) $\Rightarrow \dot{V}(X)$ Negative semi definite

What about V(X)

- There is no general method for selection of V(X).
- Some time select V(X) such that its properties are similar to energy i.e.
- $V(X) = \frac{1}{2}X^T X$
- $Or\ V(X) = Kinteic\ Energy + Potential\ Engery$
- $Or\ V(X) = x_1^2 + x_2^2$ etc
- How to calculate $\dot{V}(X)$

$$\dot{V}(X) = \left(\frac{\partial V}{\partial x}\right)^T \dot{X} = \left(\frac{\partial V}{\partial x}\right)^T f(X)$$

Lyapunov Stability Theorems

Theorem – 2 (Asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

$(i)\quad V(0) = 0$

$(ii)\ V(X) > 0,\ in\ D - \{0\}$

$(iii)\ \dot{V}(X) < 0,\ in\ D - \{0\}$

Then $X = 0$ is "asymptotically stable".

Note:

Condition (i) & (ii) $\Rightarrow$ V(X) positive definite

Condition (iii) $\Rightarrow \dot{V}(X)$ Negative definite

Lyapunov Stability Theorems

Theorem – 3 (Globally asymptotically stable)

Let $X = 0$ be an equilibrium point of $\dot{X} = f(X)$, $f : D \to \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function

such that:

(i) $V(0) = 0$

(ii) $V(X) > 0, \; in \; D - \{0\}$

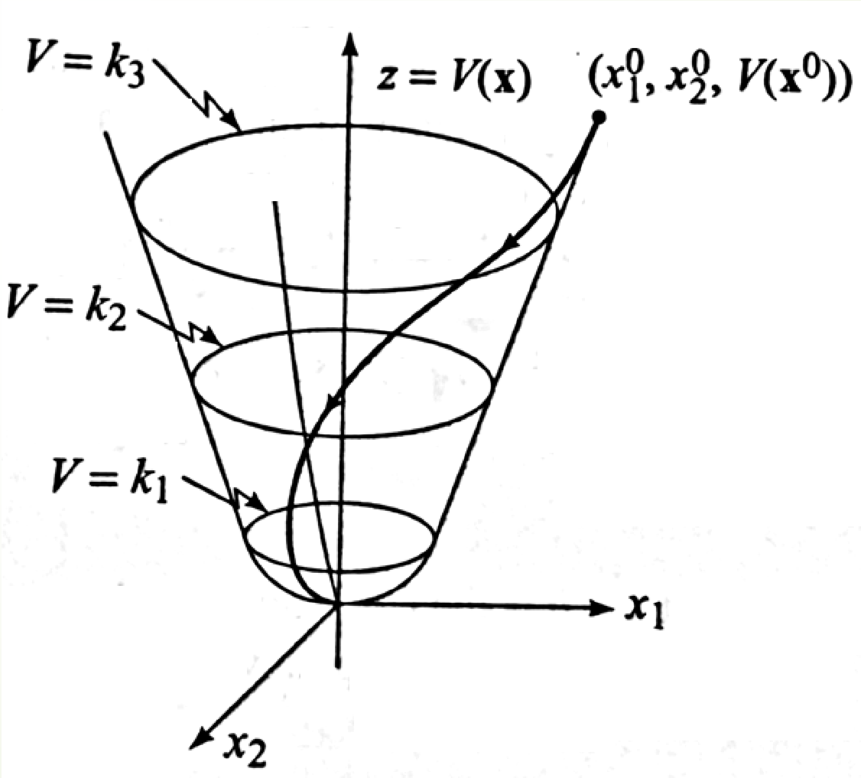
(iii) $\dot{V}(X) < 0, \; in \; D - \{0\}$

(iv) $V(X)$ is "radially unbounded"

Then $X = 0$ is "globally asymptotically stable".

Radially Unbounded ?

- The more and more you go away from the equilibrium point, V(X) will increase more and more.



Lyapunov Stability Theorems

Theorem – 3 (Exponentially stable)

Suppose all conditions for asymptotic stability are satisfied.

In addition to it, suppose $\exists$ constants k_1, k_2, k_3, p :

$$(i) \quad k_1 \|X\|^p \le V(X) \le k_2 \|X\|^p$$

$$(ii) \quad \dot{V}(X) \le -k_3 \|X\|^p$$

Then the origin $X = 0$ is "exponentially stable".

Moreover, if these conditions hold globally, then the origin $X = 0$ is "globally exponentially stable".

Note: Global⇒ Subset D=R

Example:
Pendulum Without Friction

- System dynamics $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -(g/l)\sin x_1 \end{bmatrix}$$
  $x_1 \triangleq \theta, \; x_2 \triangleq \dot{\theta}$
- Lyapunov function $$\begin{aligned} V &= KE + PE \\ &= \frac{1}{2}m(\omega l)^2 + mgh \\ &= \frac{1}{2}ml^2x_2^2 + mg(1-\cos x_1) \end{aligned}$$

Pendulum Without Friction

$$\dot{V}(X) = (\nabla V)^T f(X)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T$$

$$= \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\frac{g}{l} \sin x_1 \end{bmatrix}^T$$

$$= mglx_2 \sin x_1 - mglx_2 \sin x_1 = 0$$

$$\dot{V}(X) \leq 0 \quad (\text{nsdf})$$

Hence, it is a "stable" system.

Pendulum With Friction

Modify the previous example by adding the friction force $kl\dot{\theta}$

$$ma = -mg\sin\theta - kl\dot{\theta}$$

Defining the same state variables as above

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

Pendulum With Friction

$$\dot{V}(X) = (\nabla V)^T f(X)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(X) & f_2(X) \end{bmatrix}^T$$

$$= \begin{bmatrix} mgl \sin x_1 & ml^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 & -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{bmatrix}^T$$

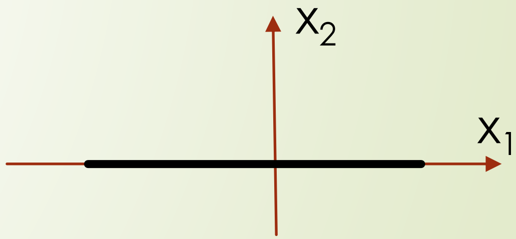
$$= -kl^2 x_2^2$$

$$\dot{V}(X) \leq 0 \quad (\text{nsdf})$$

Hence, it is also just a "stable" system.
(A frustrating result..!)

NOTE

- Here, pendulum with friction should be asymptotically stable as it comes to an equilibrium point finally due to friction ($\Rightarrow \dot{V}(X)$ **should be negative definite not negative semi definite nsdf**)
- But we are not able to prove this.
- Because
  - when $x_2 \neq 0$, $\dot{V}(X)$ **will always be –Ve**
  - **But when** $x_2 = 0$ There are multiple equilibrium points on x_1 line.
  - Negative definite means the movement I go away from the zero I should get –ve value



Example:

- Consider the system described by the equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - x_2^3$$

- **Solution:**

Choose $V(x) = x_1^2 + x_2^2$

Which satisfies following two conditions that is it is positive definite

$$V(0) = 0 \;\&\; V(x) > 0$$

$$\dot{V}(x) = 2x_1\dot{x_1} + 2x_2\dot{x_2} = 2x_1\dot{x_1} + 2x_2\left(-x_1 - x_2^3\right) = -2x_2^4$$

$\dot{V}(x) \leq 0 \Rightarrow$ nsdf (similar to pendulum with friction)

So system is stable, we can't say asymptotically stable

Analysis of LTI system using Lyapunov stability

System dynamics: $\dot{X} = AX, \quad A \in \mathbb{R}^{n\times n}$

Lyapunov function: $V(X) = X^T P X, \qquad P > 0 \ (\text{pdf})$

Derivative analysis:

$$\begin{aligned} \dot{V} &= \dot{X}^T P X + X^T P \dot{X} \\ &= X^T A^T P X + X^T P A X \\ &= X^T \left(A^T P + P A \right) X \end{aligned}$$

Note: $\dot{X} = AX \Rightarrow \dot{X}^T = (AX)^T = X^T A^T$

Analysis of LTI system using Lyapunov stability…

For stability, we aim for $\dot{V} = -X^T Q X \quad (Q > 0)$

By comparing $X^T \left(A^T P + PA \right) X = -X^T Q X$

For a non-trivial solution

$$\boxed{PA + A^T P + Q = 0}$$

(Lyapunov Equation)

Analysis of LTI system using Lyapunov stability….

Theorem : The eigenvalues λ_i of a matrix $A \in \mathbb{R}^{n \times n}$ satisfy $\mathrm{Re}(\lambda_i) < 0$ if and only if for any given symmetric *pdf* matrix Q, $\exists$ a unique *pdf* matrix P satisfying the Lyapunov equation.

Step to solve
Analysis of LTI system using Lyapunov stability….

- Choose an arbitrary symmetric positive definite matrix Q $(Q = I)$
- Solve for the matrix P form the *Lyapunov equation* and verify whether it is positive definite
- Result: If P is positive definite, then $\dot{V}(X) < 0$ and hence the origin is “asymptotically stable”.

Example: Analysis of LTI system using Lyapunov stability

- Determine the stability of the system described by the following equation
- $\dot{x} = Ax$ With $A = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}$
- Solution:
- $$A^T P + PA = -Q = -I$$
- $$\begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} p11 & p12 \\ p12 & p22 \end{bmatrix} + \begin{bmatrix} p11 & p12 \\ p12 & p22 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
- Note here we took p12=p21 because Matrix P will be + real symmetric matrix

- $-2p_{11}+2p_{12}=-1$
- $-2p_{11}-5p_{12}+p_{22}=0$
- $-4p_{12}-8p_{22}=-1$
- Solving above three equations $P = \begin{bmatrix} p11 & p12 \\ p12 & p22 \end{bmatrix} = \begin{bmatrix} \frac{23}{60} & -\frac{7}{60} \\ -\frac{7}{60} & \frac{11}{60} \end{bmatrix}$
- which is seen to be positive definite. Hence this system is asymptotically stable

Till now ?

- All were **Lyapunov Direct methods**
- There are some **indirect methods** also

Lyapunov's Indirect Theorem

Let the linearized system about $X = 0$ be $\Delta\dot{X} = A(\Delta X)$.

The theorem says that if all the eigenvalues λ_i $(i = 1, \ldots, n)$ of the matrix A satisfy $\mathrm{Re}(\lambda_i) < 0$ (i.e. the linearized system is exponentially stable), then for the nonlinear system the origin is locally exponentially stable.

Instability theorem

Consider the autonomous dynamical system and assume $X=0$ is an equilibrium point. Let $V : D \to \mathbb{R}$ have the following properties:

$(i)\ V(0) = 0$

$(ii)\ \exists X_0 \in \mathbb{R}^n$, arbitrarily close to $X = 0$, such that $V(X_0) > 0$

$(iii)\ \dot{V} > 0 \quad \forall X \in U$, where the set U is defined as follows

$$U = \{X \in D : \|X\| \leq \varepsilon \ \text{and} \ V(X) > 0\}$$

Under these conditions, $X=0$ is unstable

In rough way

- In rough way instability theorem state that
  - if $V(X)$ positive definite
  - then $\dot{V}(X)$ should also be positive definite

Positive Definite Matrices

Definition: Symmetric matrix $M = M^T$ is

- **positive definite** $(M > 0)$ if $x^T M x > 0, \ \forall x \neq 0$
- **positive semidefinite** $(M \geq 0)$ if $x^T M x \geq 0, \ \forall x$

Thanks

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