Construction of Lyapunov function

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Construction of Lyapunov function

- **Major issues**
  - 1. If above to methods i.e. $\boldsymbol{V(X) = \frac{1}{2}X^T X}$ $Or\ V(X) = Kinteic\ Energy + Potential\ Engery$ fails, what should we do now? Is there any standard procedure to find out the Lyapunov fucnion ? Answer to this is yes
  - 2. If system is nsdf then there may be issues related to system whether it is stable or asymptotically stable.
- **There are two methods to construct the Lyapunov function**
  - 1. By Variable Gradient method
  - 2. By Krasovskii's Method

Variable Gradient method

- We know $\dot{V}(X) = \left(\frac{\partial V}{\partial x}\right)^T \dot{X} = \left(\frac{\partial V}{\partial x}\right)^T f(X)$
- **Instead of selecting $\dot{V}(X)$ we will select $\Delta V = \frac{\partial V}{\partial x}$ and then find out condition for V(X)**

Variable Gradient method…

* Select a $\nabla V = \dfrac{\partial V}{\partial X} = g(X)$ that contains some adjustable parameters

* Then $dV(X) = \left(\dfrac{\partial V}{\partial X}\right)^T dX$

$$\int_{\tilde{X}=0}^{X} dV(\tilde{X}) = \int_{\tilde{X}=0}^{X} \left(\frac{\partial V}{\partial \tilde{X}}\right)^T d\tilde{X}$$

$$V(X) - V(0) = \int_{\tilde{X}=0}^{X} g(\tilde{X})\, d\tilde{X}$$

0

Note:

To recover a unique V, $\nabla V = g(X)$ must satisfy the "Curl Condition":

i.e. $\dfrac{\partial g_i}{\partial x_j} = \dfrac{\partial g_j}{\partial x_i}$

However, note that the intergal value depends on the initial and final states (not on the path followed). Hence, integration can be conveniently done along each of the co-ordinate axes in turn; i.e.

Variable Gradient method…

$$V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots\ldots, 0)\, d\tilde{x}_1$$
$$+ \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2$$
$$\vdots$$
$$+ \int_0^{x_n} g_n(x_1, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n$$

Note: The free parameter of $g(X)$ are constrained to satisfy the symmetric condition, which is satisfied by all gradients of a scalar functions.

- Curl Condition means
$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$
This will result $\frac{\partial g}{\partial x}$ matrix as a symmetrical matrix
- integration can be conveniently done along each of the co-ordinate axes means, we can move first along x_1 axis than x_2 axis than x_3 & so on that means we can integrate first w.r.t x_1 than w.r.t. x_2 & so on

Variable Gradient method…

Theorem: A function $g(X)$ is the gradient of a scalar function $V(X)$ if and only if the matrix $\left[\dfrac{\partial g(X)}{\partial X}\right]$ is symmetric; where

$$\left[\frac{\partial g(X)}{\partial X}\right] \triangleq \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{bmatrix}$$

Proof: Please see Marquez book (Appendix)

Variable Gradient method…

Proof : (Necessity)

Assume: $g(X) = \dfrac{\partial V}{\partial X}$

$$\frac{\partial g(X)}{\partial X} = \frac{\partial^2 V}{\partial X^2}$$

$$= \begin{pmatrix} \dfrac{\partial^2 V}{\partial x_1{}^2} & \dfrac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & & \vdots \\ \dfrac{\partial^2 V}{\partial x_n \partial x_1} & \dfrac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 V}{\partial x_n^2} \end{pmatrix}$$

Variable Gradient method…

$$\because \quad \frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} \Rightarrow \boxed{\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}}$$

Hence, the matrix $\left[\frac{\partial g(X)}{\partial X}\right]$ should be symmetric.

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}$$

⇒Take partial derivative in any sequence

Variable Gradient method…

Sufficiency: Assume $\dfrac{\partial g_i}{\partial x_j} = \dfrac{\partial g_j}{\partial x_i}$

$$\left[\text{To show } \frac{\partial V}{\partial x_i} = g_i(X) \quad \forall i\right]$$

Variable Gradient method…

We have:

$$V(X) = \int_0^x g(\tilde{x})\, d\tilde{x}$$

$$= \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots\ldots, 0)\, d\tilde{x}_1$$

$$+ \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\, d\tilde{x}_2$$

$$+ \int_0^{x_n} g_n(x_1, x_2, \ldots\ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n$$

Take partial derivate of V(X) w.r.to x_1

Variable Gradient method…

$$\frac{\partial V}{\partial x_1} = g_1(x_1, 0, \ldots\ldots, 0)$$

$$+ \int_0^{x_2} \frac{\partial g_2}{\partial x_1}(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\,d\tilde{x}_2$$

$$\vdots$$

$$+ \int_0^{x_n} \frac{\partial g_n}{\partial x_1}(x_1, x_2, \ldots\ldots, x_{n-1}, \tilde{x}_n)\,d\tilde{x}_n$$

$$= g_1(x_1, 0, \ldots\ldots, 0) + \int_0^{x_2} \frac{\partial g_1}{\partial x_2}(x_1, \tilde{x}_2, 0, \ldots\ldots, 0)\,d\tilde{x}_2 + \ldots$$

$$+ \int_0^{x_n} \frac{\partial g_1}{\partial x_n}(x_1, x_2, \ldots\ldots, x_{n-1}, \tilde{x}_n)\,d\tilde{x}_n$$

$using$

$$\frac{\partial g_2}{\partial x_1} = \frac{\partial g_1}{\partial x_2}$$

And so on

Variable Gradient method…

$$
\begin{aligned}
= \quad & g_1(x_1,0,\ldots\ldots,0) + g_1(x_1,\tilde{x}_2,0,\ldots\ldots,0)\Big|_{\tilde{x}_2=0}^{x_2} \\
& + \cdots + g_1(x_1,x_2,\ldots\ldots,x_{n-1},\tilde{x}_n\,)\Big|_{\tilde{x}_n=0}^{x_n} \\
= \quad & \cancel{g_1(x_1,0,\ldots\ldots,0)} + \left[g_1(x_1,\tilde{x}_2,0,\ldots\ldots,0) - \cancel{g_1(x_1,0,\ldots\ldots,0)}\right] \\
& + \cdots + \left[\underbrace{g_1(x_1,x_2,\ldots\ldots,x_n)} - g_n(x_1,x_2,\ldots\ldots,x_n,0)\right] \\
= \quad & g_1(x_1,x_2,\ldots\ldots,x_n)
\end{aligned}
$$

i.e $$\boxed{\frac{\partial V}{\partial x_1} = g_1(X)}$$

Similarly $$\frac{\partial V}{\partial x_i} = g_i(X) \quad , \quad \forall i = 1,\cdots,n$$

Rest terms will cancel out

Variable Gradient Method: Example

<u>Problem</u>: Analyze the stability behaviour of the following system

$$\dot{x}_1 = -ax_1$$
$$\dot{x}_2 = bx_2 + x_1 x_2^2$$

<u>Solution</u>: $X = 0$ is an equilibrium point

$$\text{Assume } \frac{\partial V}{\partial X} = g(X) = \underbrace{\begin{pmatrix} k_1 & k \\ k & k_2 \end{pmatrix}}_{\text{A symmetric matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\text{Note:} \quad \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} = k \right)$$

Note: For equilibrium point

$$\dot{x}_1 = -ax_1 = 0$$
$$\dot{x}_2 = bx_2 + x_1 x_2^2 = 0$$
$$\Rightarrow x_1 = 0 \ \& \ x_2 = 0$$
$$\Rightarrow X = 0$$

Variable Gradient Method: Example

Further, let us assume

$$\therefore \quad \frac{\partial V}{\partial X} = \begin{bmatrix} g_1(X) \\ g_2(X) \end{bmatrix} = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \end{bmatrix}$$

Taking k=0⇒ g(x) will be diagonal symmetrical matrix

$$\Rightarrow \quad V(X) = \int_0^{x_1} g_1(\tilde{x}_1, 0)\, d\tilde{x}_1 + \int_0^{x_2} g_2(x_1, \tilde{x}_2)\, d\tilde{x}_2$$

$$= \int_0^{x_1} k_1 \tilde{x}_1 d\tilde{x}_1 + \int_0^{x_2} k_2 \tilde{x}_2 d\tilde{x}_2$$

$$= \frac{1}{2}\left(k_1 x_1^{\,2} + k_2 x_2^{\,2}\right)$$

Variable Gradient Method:

Choose $\boxed{k_1, k_2 > 0}$

Then $V(X) > 0 \quad \forall X \neq 0$ and $V(0) = 0$

$V(X)$ is a Lyapunov function candidate.

$$\dot{V}(X) = g^T(X) f(X) = \begin{bmatrix} k_1 x_1 & k_2 x_2 \end{bmatrix} \begin{bmatrix} -a x_1 \\ b x_2 + x_1 x_2{}^2 \end{bmatrix}$$

$$= -k_1 a x_1^2 + k_2 (b + x_1 x_2) x_2^2$$

Let us choose $k_1 = k_2 = 1$. Then

$$\dot{V}(X) = -a x_1^2 + (b + x_1 x_2) x_2^2$$

Variable Gradient Method:

Unless we know about a, b at this point nothing can be said about $\dot{V}(X)$. Let us assume $a > 0,\ b < 0$. Then

$$\dot{V}(X) = -ax_1^2 - \underbrace{\left(|b| - x_1x_2\right)}_{>0 \text{ (for small } x_1x_2\text{)}} x_2^2$$

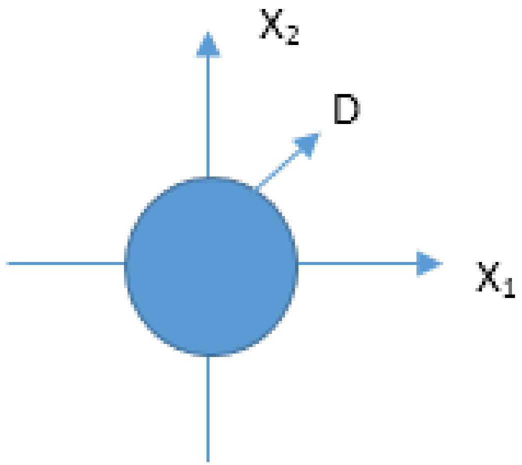
$\therefore \quad \dot{V}(X) < 0$ in some domain $D \subset \mathbb{R}^2$ and $0 \in D$

i.e $\dot{V}(X)$ is negative definite in D

$\therefore$ The system is <u>locally asymptotically stable!</u>

Variable gradient method…

- Explanation



⇒There exist a small domain D which include equilibrium point, where sys is asymptotically stable

Krasovskii's Method

Let us consider the system $\dot{X} = f(X)$

Let $A(X) \triangleq \left[\dfrac{\partial f}{\partial X}\right]$: Jacobian matrix

Theorem :

If the matrix $F(X) \triangleq A(X) + A^T(X)$ is ndf for all $X \in D \quad (0 \in D)$,

then the equilibrium point is locally asymptotically stable and a

Lyapunov function for the system is

$$V(X) = f^T(X) f(X)$$

Note: If $D = \mathbb{R}^n$ and $V(X)$ is radially unbounded,

then the equilibrium point is globally asymptotically stable.

Krasovskii's Method

$$
\begin{aligned}
\dot{V}(X) &= f^T \dot{f} + \dot{f}^T f \\
&= f^T \left[\frac{\partial f}{\partial X}\right]^T \dot{X} + \dot{X}^T \left[\frac{\partial f}{\partial X}\right] f \\
&= f^T \left(A^T + A\right) f \\
&= f^T F\, f
\end{aligned}
$$

Hence, if $F(X)$ is negative definite, $\dot{V}(X)$ is ndf.

So, by Lyapunov's theorem, $X = 0$ is asymptotically stable.

Krasovskii's Method

Note: The global asymptotic stability of the system is guaranteed by the Global version of Lyapunov's direct method.

Comment: While the usage of this result is fairly straight forward, its applicability is limited in practice since $F(X)$ for many systems do not satisfy the negative definite property.

Generalized Krasovskii's Theorem

Theorem :

Let $\quad A(X) \triangleq \left[\dfrac{\partial f(X)}{\partial X}\right]$

A sufficent condition for the origin to be asymptotically stable is that

$\exists$ two pdf matrices P and Q: $\forall X \neq 0$, the matrix

$$F(X) = A^T P + PA + Q$$

is negative semi-definite in some neighbourhood D of the origin.

In addition, if $D = \mathbb{R}^n$ and $V(X) \triangleq f^T(X)\, P\, f(X)$ is radially unbounded, then the system is globally asymptotically stable.

Generalized Krasovskii's Theorem

Proof : $V(X) = f^T(X)Pf(X)$

$$\dot{V}(X) = \left[f^T P \dot{f} + \dot{f}^T P f\right]$$

$$= f^T P\left(\frac{\partial f}{\partial X}\right)^T \dot{X} + \left[\left(\frac{\partial f}{\partial X}\right)^T \dot{X}\right]^T Pf$$

$$= f^T PA^T f + f^T AP f$$

$$= f^T\left(PA^T + AP + Q - Q\right)f$$

$$= \underbrace{f^T\left(PA^T + AP + Q\right)f}_{nsdf} - \underbrace{f^T Qf}_{ndf}$$

< 0 (ndf) Hence, the result.

Example

Problem: Analyze the stability behaviour of the following system

$$\dot{x}_1 = -6x_1 + 2x_2$$

$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^{\,3}$$

Solution:

$$A = \left[\frac{\partial f}{\partial X}\right] = \begin{bmatrix} -6 & 2 \\ 2 & -6-6x_2^{\,2} \end{bmatrix}$$

$$F = A + A^T = \begin{bmatrix} -12 & 4 \\ 4 & -12-12x_2^{\,2} \end{bmatrix}$$

Example

Eigen values of F:

$$\begin{vmatrix} \lambda+12 & -4 \\ -4 & \lambda+12+12x_2^{\,2} \end{vmatrix} = 0$$

$$(\lambda+12)^2 + (\lambda+12)12x_2^{\,2} - 16 = 0$$

$$\lambda^2 + 24\lambda + 144 + 12x_2^{\,2}\lambda + 144x_2^{\,2} - 16 = 0$$

$$\lambda^2 + \left(24+12x_2^{\,2}\right)\lambda + \left(128+144x_2^{\,2}\right) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-\left(24+12x_2^{\,2}\right) \pm \sqrt{\left(24+12x_2^{\,2}\right)^2 - 4\left(128+144x_2^{\,2}\right)}\right]$$

Example

$$= -(12+6x_2^{\,2}) \pm \underbrace{\sqrt{\left(12+6x_2^{\,2}\right)^2 - \left(128+144x_2^{\,2}\right)}}_{0<(*)<(12+6x_2^2)}$$

$$< 0 \qquad \forall\, x_2 \in \mathbb{R}$$

$$\therefore \quad \text{A is ndf in } \mathbb{R}^2$$

Morever, $V(X) = f^T(X)\, f(X)$

$$= \left(-6x_1 + 2x_2\right)^2 + \left(2x_1 - 6x_2 - 2x_2^{\,3}\right)^2$$

$$\rightarrow \infty \text{ as } \|X\| \rightarrow \infty$$

$\therefore \quad X = 0$ is globally asymptotically stable.

Thanks

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