Construction of Lyapunov function

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- **Major issues**
  - 1. If above to methods i.e. $V(X) = \frac{1}{2}X^TX$ Or $V(X) = Kinear Energy + Potential Energy$ fails, what should we do now? Is there any standard procedure to find out the Lyapunov function? Answer to this is yes.
  - 2. If system is nsdf then there may be issues related to system whether it is stable or asymptotically stable.

- **There are two methods to construct the Lyapunov function**
  - 1. By Variable Gradient method
  - 2. By Krasovskii’s Method
Variable Gradient method

- We know $\dot{V}(X) = \left( \frac{\partial V}{\partial x} \right)^T \dot{X} = \left( \frac{\partial V}{\partial x} \right)^T f(X)$

- Instead of selecting $\dot{V}(X)$ we will select $\Delta V = \frac{\partial V}{\partial x}$ and then find out condition for $V(X)$
Variable Gradient method...

* Select a $\nabla V = \frac{\partial V}{\partial X} = g(X)$ that contains some adjustable parameters

* Then $dV(X) = \left( \frac{\partial V}{\partial X} \right)^T dX$

$$\int_{\tilde{X}_0}^{X} dV(\tilde{X}) = \int_{\tilde{X}_0}^{X} \left( \frac{\partial V}{\partial \tilde{X}} \right)^T d\tilde{X}$$

$$V(X) - V(0) = \int_{\tilde{X}_0}^{X} g(\tilde{X}) d\tilde{X}$$

Note:
To recover a unique $V$, $\nabla V = g(X)$ must satisfy the "Curl Condition":

i.e. $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$

However, note that the integral value depends on the initial and final states (not on the path followed). Hence, integration can be conveniently done along each of the co-ordinate axes in turn; i.e.
Variable Gradient method...

\[ V(X) = \int_{0}^{x_1} g_1(\tilde{x}_1, 0, \ldots, 0)d\tilde{x}_1 + \int_{0}^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots, 0)d\tilde{x}_2 + \cdots + \int_{0}^{x_n} g_n(x_1, \ldots, x_{n-1}, \tilde{x}_n)d\tilde{x}_n \]

Note: The free parameter of \( g(X) \) are constrained to satisfy the symmetric condition, which is satisfied by all gradients of a scalar functions.

- Curl Condition means
  \[ \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \]
  This will result \( \frac{\partial g}{\partial x} \) matrix as a symmetrical matrix.
  - integration can be conveniently done along each of the co-ordinate axes means, we can move first along \( x_1 \) axis than \( x_2 \) axis than \( x_3 \) & so on that means we can integrate first w.r.t \( x_1 \) than w.r.t. \( x_2 \) & so on.
Variable Gradient method...

Theorem: A function \( g(X) \) is the gradient of a scalar function \( V(X) \) if and only if the matrix \( \frac{\partial g(X)}{\partial X} \) is symmetric, where

\[
\frac{\partial g(X)}{\partial X} = \begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\
\frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_2}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n}
\end{bmatrix}
\]

Proof: Please see Marquez book (Appendix)
Variable Gradient method...

**Proof : (Necessity)**

Assume:

\[ g(X) = \frac{\partial V}{\partial X} \]

\[ \frac{\partial g(X)}{\partial X} = \frac{\partial^2 V}{\partial X^2} \]

\[ = \begin{pmatrix}
\frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\
\vdots & \ddots & & \vdots \\
\frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2}
\end{pmatrix} \]
Variable Gradient method...

\[
\begin{align*}
\therefore \quad \frac{\partial^2 V}{\partial x_i \partial x_j} &= \frac{\partial^2 V}{\partial x_j \partial x_i} \\
\text{Hence, the matrix } \begin{bmatrix} \frac{\partial g(X)}{\partial X} \end{bmatrix} \text{ should be symmetric.}
\end{align*}
\]

⇒ Take partial derivative in any sequence
Variable Gradient method...

Sufficiency: Assume \( \frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \)

To show \( \frac{\partial V}{\partial x_i} = g_i(X) \quad \forall i \)
Variable Gradient method...

We have:

\[ V(X) = \int_0^{x_1} g(\tilde{x}) \, d\tilde{x} \]

\[ = \int_0^{x_1} g_1(\tilde{x}_1, 0, \ldots, 0) \, d\tilde{x}_1 \]

\[ + \int_0^{x_2} g_2(x_1, \tilde{x}_2, 0, \ldots, 0) \, d\tilde{x}_2 \]

\[ + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) \, d\tilde{x}_n \]

Take partial derivative of \( V(X) \) w.r.t. \( x_1 \)
Variable Gradient method...

\[ \frac{\partial V}{\partial x_1} = g_1(x_1, 0, \ldots, 0) \]

\[ + \int_0^{x_2} \frac{\partial g_2}{\partial x_1}(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 \]

\[ \vdots \]

\[ + \int_0^{x_n} \frac{\partial g_n}{\partial x_1}(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n \]

\[ = g_1(x_1, 0, \ldots, 0) + \int_0^{x_2} \frac{\partial g_1}{\partial x_2}(x_1, \tilde{x}_2, 0, \ldots, 0) d\tilde{x}_2 + \ldots \]

\[ + \int_0^{x_n} \frac{\partial g_1}{\partial x_n}(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) d\tilde{x}_n \]

using \[ \frac{\partial g_2}{\partial x_1} = \frac{\partial g_1}{\partial x_2} \]

And so on
Variable Gradient method...

\[
\begin{align*}
= & \ g_1(x_1, 0, \ldots, 0) + g_1(x_1, \tilde{x}_2, 0, \ldots, 0) \bigg|_{\tilde{x}_2=0} \\
+ & \cdots + g_1(x_1, x_2, \ldots, x_{n-1}, \tilde{x}_n) \bigg|_{\tilde{x}_n=0} \\
= & \ g_1(x_1, 0, \ldots, 0) + \left[ g_1(x_1, \tilde{x}_2, 0, \ldots, 0) - g_1(x_1, 0, \ldots, 0) \right] \\
+ & \cdots + \left[ g_1(x_1, x_2, \ldots, x_n) - g_n(x_1, x_2, \ldots, x_n, 0) \right] \\
= & \ g_1(x_1, x_2, \ldots, x_n)
\end{align*}
\]

i.e

\[
\frac{\partial V}{\partial x_1} = g_1(X)
\]

Similarly

\[
\frac{\partial V}{\partial x_i} = g_i(X), \quad \forall i = 1, \ldots, n
\]

Rest terms will cancel out
Variable Gradient Method: Example

Problem: Analyze the stability behaviour of the following system

\[
\begin{align*}
\dot{x}_1 &= -ax_1 \\
\dot{x}_2 &= bx_2 + x_1x_2^2
\end{align*}
\]

Solution: \( X = 0 \) is an equilibrium point

\[
\frac{\partial V}{\partial X} = g(X) = \begin{pmatrix} k_1 & k \\ k & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

A symmetric matrix

\[
\begin{pmatrix}
\text{Note: } \frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} = k
\end{pmatrix}
\]

Note: For equilibrium point

\[
\begin{align*}
\dot{x}_1 &= -ax_1 = 0 \\
\dot{x}_2 &= bx_2 + x_1x_2^2 = 0
\end{align*}
\]

\( \Rightarrow x_1 = 0 \) & \( x_2 = 0 \)

\( \Rightarrow X = 0 \)
Variable Gradient Method: Example

Further, let us assume

\[
\begin{align*}
\therefore \quad \frac{\partial V}{\partial X} &= \begin{bmatrix} g_1(X) \\ g_2(X) \end{bmatrix} = \begin{bmatrix} k_1 x_1 \\ k_2 x_2 \end{bmatrix} \\
\Rightarrow \quad V(X) &= \int_0^{x_1} g_1(\tilde{x}_1, 0) \, d\tilde{x}_1 + \int_0^{x_2} g_2(x_1, \tilde{x}_2) \, d\tilde{x}_2 \\
&= \int_0^{x_1} k_1 \tilde{x}_1 \, d\tilde{x}_1 + \int_0^{x_2} k_2 \tilde{x}_2 \, d\tilde{x}_2 \\
&= \frac{1}{2} \left( k_1 x_1^2 + k_2 x_2^2 \right)
\end{align*}
\]

Taking \(k=0\) \(\Rightarrow g(x)\) will be diagonal symmetrical matrix
Variable Gradient Method:

Choose \[ k_1, k_2 > 0 \]

Then \( V(X) > 0 \quad \forall X \neq 0 \) and \( V(0) = 0 \)

\( V(X) \) is a Lyapunov function candidate.

\[
\dot{V}(X) = g^T(X)f(X) = \begin{bmatrix} k_1x_1 & k_2x_2 \end{bmatrix} \begin{bmatrix} -ax_1 \\ bx_2 + x_1x_2^2 \end{bmatrix}
\]

\[
= -k_1ax_1^2 + k_2(b + x_1x_2)x_2^2
\]

Let us choose \( k_1 = k_2 = 1 \). Then

\[
\dot{V}(X) = -ax_1^2 + (b + x_1x_2)x_2^2
\]
Variable Gradient Method:

Unless we know about $a, b$ at this point nothing can be said about $\dot{V}(X)$. Let us assume $a > 0, \ b < 0$. Then

$$\dot{V}(X) = -ax_1^2 - (|b| - x_1x_2) x_2^2$$

$> 0$ (for small $x_1x_2$)

$\therefore \dot{V}(X) < 0$ in some domain $D \subset \mathbb{R}^2$ and $0 \in D$

i.e $\dot{V}(X)$ is negative definite in $D$

$\therefore$ The system is **locally asymptotically stable!**
Variable gradient method...

- Explanation

\[ \Rightarrow \text{There exist a small domain } D \text{ which include equilibrium point, where sys is asymptotically stable} \]
Krasovskii’s Method

Let us consider the system \( \dot{X} = f(X) \)

Let \( A(X) \triangleq \begin{bmatrix} \frac{\partial f}{\partial X} \end{bmatrix} \) : Jacobian matrix

**Theorem:**
If the matrix \( F(X) \triangleq A(X) + A^T(X) \) is ndf for all \( X \in D \) \((0 \in D)\), then the equilibrium point is locally asymptotically stable and a Lyapunov function for the system is

\[ V(X) = f^T(X)f(X) \]

**Note:** If \( D = \mathbb{R}^n \) and \( V(X) \) is radially unbounded, then the equilibrium point is globally asymptotically stable.
Krasovskii’s Method

\[ \dot{V}(X) = f^T \dot{f} + \dot{f}^T f \]

\[ = f^T \left[ \frac{\partial f}{\partial X} \right]^T \dot{X} + \dot{X}^T \left[ \frac{\partial f}{\partial X} \right] f \]

\[ = f^T (A^T + A) f \]

\[ = f^T F f \]

Hence, if \( F(X) \) is negative definite, \( \dot{V}(X) \) is ndf.

So, by Lyapunov's theorem, \( X = 0 \) is asymptotically stable.
Krasovskii’s Method

**Note:** The global asymptotic stability of the system is guaranteed by the Global version of Lyapunov's direct method.

**Comment:** While the usage of this result is fairly straightforward, its applicability is limited in practice since $F(X)$ for many systems do not satisfy the negative definite property.
Generalized Krasovskii’s Theorem

Theorem:

Let \[ A(X) \triangleq \begin{bmatrix} \frac{\partial f(X)}{\partial X} \end{bmatrix} \]

A sufficient condition for the origin to be asymptotically stable is that

\[ \exists \text{ two pdf matrices } P \text{ and } Q: \ \forall X \neq 0, \text{ the matrix} \]

\[ F(X) = A^T P + PA + Q \]

is negative semi-definite in some neighbourhood \( D \) of the origin.

In addition, if \( D = \mathbb{R}^n \) and \( V(X) \triangleq f^T(X) P f(X) \) is radially unbounded, then the system is globally asymptotically stable.
Generalized Krasovskii’s Theorem

**Proof:**

\[
V(X) = f^T(X) Pf(X)
\]

\[
\dot{V}(X) = [f^T P \dot{f} + \dot{f}^T P f]
\]

\[
= f^T P \left( \frac{\partial f}{\partial X} \right)^T \dot{X} + \left( \frac{\partial f}{\partial X} \right)^T \dot{X} \right]^T P f
\]

\[
= f^T PA^T f + f^T AP f
\]

\[
= f^T \left( PA^T + AP + Q - Q \right) f
\]

\[
= f^T \left( \underbrace{PA^T}_{nsdf} + AP + Q \right) f - f^T Q f
\]

\[
< 0 \quad (ndf)
\]

Hence, the result.
Example

Problem: Analyze the stability behaviour of the following system
\[
\begin{align*}
\dot{x}_1 &= -6x_1 + 2x_2 \\
\dot{x}_2 &= 2x_1 - 6x_2 - 2x_2^3
\end{align*}
\]

Solution:
\[
A = \begin{bmatrix}
\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
-6 & 2 \\
2 & -6 - 6x_2^2
\end{bmatrix}
\]
\[
F = A + A^T = \begin{bmatrix}
-12 & 4 \\
4 & -12 - 12x_2^2
\end{bmatrix}
\]
Example

Eigen values of $F$:

\[
\begin{vmatrix}
\lambda + 12 & -4 \\
-4 & \lambda + 12 + 12x_2^2
\end{vmatrix} = 0
\]

\[
(\lambda + 12)^2 + (\lambda + 12)12x_2^2 - 16 = 0
\]

\[
\lambda^2 + 24\lambda + 144 + 12x_2^2\lambda + 144x_2^2 - 16 = 0
\]

\[
\lambda^2 + \left(24 + 12x_2^2\right)\lambda + \left(128 + 144x_2^2\right) = 0
\]

\[
\lambda_{1,2} = \frac{1}{2} \left[ -\left(24 + 12x_2^2\right) \pm \sqrt{\left(24 + 12x_2^2\right)^2 - 4\left(128 + 144x_2^2\right)} \right]
\]
Example

\[-(12 + 6x_2^2) \pm \sqrt{(12 + 6x_2^2)^2 - (128 + 144x_2^2)}\]

\[< 0 \quad \forall \ x_2 \in \mathbb{R}\]

\[\therefore \ A \text{ is ndf in } \mathbb{R}^2\]

Moreover,

\[V(X) = f^T(X)f(X)\]

\[= (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2\]

\[\rightarrow \infty \text{ as } \|X\| \rightarrow \infty\]

\[\therefore X = 0 \text{ is globally asymptotically stable.}\]
Thanks

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